CHAPTER 9

Calculus Relationships in AP Physics C: Electricity and Magnetism

This chapter focuses on some of the quantitative skills that are important in your AP Physics C: Mechanics course. These are not all of the skills that you will learn, practice, and apply during the year, but these are the skills you will most likely encounter as part of your laboratory investigations or classroom experiences, and potentially on the AP Physics C Exam.

Electrostatics: Electric Fields

Coulomb's Law

The fundamental law of electrostatics is Coulomb's law. This law describes the interaction between two independent charges. All charges interact with all other charges through a distance. Like charges will repel and unlike charges will attract: this defines the direction of the forces on each charge by the other charge. Coulomb's law describes how to compute the magnitude of force that each charge exerts on the other charge.

$$\left|\vec{F}\right| = \frac{1}{4\pi\varepsilon_0}\frac{\left|q_1 q_2\right|}{r^2}$$

Example

Two positive static charges are held fixed in space and separated by a distance of $r = 0.5m$. The first charge has a magnitude of $q_1 = 1.0\times10^{-6}\,C$ and the second charge has a magnitude of $q_2 = 2.0\times10^{-6}\,C$. Determine the magnitude of force of q_1 acting on q_2.

The permittivity of free space is defined as

$$\varepsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{Nm^2}$$

and the quantity

$$\frac{1}{4\pi\varepsilon_0} = 9.0\times10^9\,\frac{Nm^2}{C^2}$$

is also called "k" or Coulomb's constant.

The computation for the magnitude of the force is

$$\left|\vec{F}\right| = \frac{(9\times10^{9}\frac{Nm^{2}}{C^{2}})(1.0\times10^{-6}C)(2.0\times10^{-6}C)}{(0.5m)^{2}} = 7.2\times10^{-2}N$$

If you need more information, the following tutorial can help to further explain this concept:

Khan Academy: Coulomb's law

Definition of Electric Field

The electric field is a physical quantity defined as the ratio of the electrostatic force to the magnitude of charge that is experiencing the force. The electric field is a vector quantity and is defined as the direction that a positive test charge would move in if placed in the field:

$$\vec{E} = \frac{\vec{F}}{q'}$$

where q' is called a test charge — the charge that experiences the force.

Example

A test charge of $q' = 1.0x10^{-9}C$ experiences a force of magnitude $2.5x10^{-6}N$ when it is placed within a uniform electric field. Determine the magnitude of the electric field.

$$\left|\vec{E}\right| = \frac{\left|\vec{F}\right|}{q'} = \frac{2.5x10^{-6}N}{1.0x10^{-9}C} = 2.5x10^{3}\frac{N}{C}$$

General Definition of Flux

Flux is defined qualitatively as the magnitude of a vector field that permeates space through a particular defined area. Let us define any vector field as $\vec{T}$. The precise mathematical definition is

$$\phi = \int \vec{T}\cdot d\vec{A}$$

where ϕ is defined as flux.

Note that flux is a scalar quantity that comes from two vector quantities. In using flux as a physical quantity in physics, we need to define the area of some geometrical shape as having a vector orientation that is perpendicular (and outward) from the shape or object. So, in the case of a piece of paper flat on a desk, the area of that piece of paper has a magnitude of 8.5" × 11" and a direction of vertically upward from the piece of paper (perpendicular to the paper). The upward direction is arbitrary, but when the area is attached to an actual object the direction is defined to be outward from the object. Now that we have defined flux, we can see the specific physical definitions of flux that exist in our physics course.

Electric Flux

Electric flux is defined qualitatively as the magnitude of the electric field that permeates space through a particular defined area. The precise mathematical definition is

$$\phi_E = \int \vec{E} \cdot d\vec{A}$$

It is important to note that flux is a **scalar** quantity and is computed from two vector quantities using the vector dot product. It is also important to note that the area vector of a defined area is a vector that is perpendicular to the area's face and directed outward from the surface.

Electric Flux and Gauss's Law

Gauss's law is a fundamental law of electrostatics that relates the electric flux through a closed surface to a physical constant of the electrostatics system. Gauss's law states that the electric flux through a closed imaginary surface (known as a Gaussian surface) is proportional to the charge enclosed by the imaginary surface.

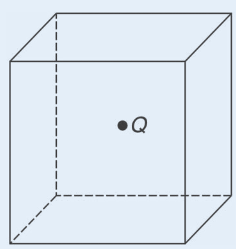
$$\oint \vec{E} \cdot d\vec{A} = \frac{q_E}{\varepsilon_0}$$

The law does make use of what is called a surface integral, but in order for the law to be useful (to determine unknown electric fields of different charge configurations), no actual integration is necessary. So a very complex-looking calculus expression is actually a very powerful and subtle conceptual law.

Gauss's law is a difficult law to grasp for most physics students. It typically takes a few weeks and many practice examples, situations, and interesting physical problems to master.

Example

An isolated point charge of magnitude $+Q$ is shown in the center of a metal cube.



Determine the electric flux through the entire cube.

The flux can be determined simply by knowing that the charge is enclosed by the closed surface — no need for integration or understanding how to compute a surface integral. Therefore, the flux through the cube is

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

$$\therefore \phi = \frac{Q}{\varepsilon_0}$$

The flux is a positive value due to the field of the point charge going out of the six faces. The vector product of the area faces and the electric field give a positive value.

What will happen to the electric flux through the cube if the cube shown above is increased to three times the size of the original cube and the same magnitude of charge (+*Q*) is still located at the center? Will the flux increase to three times the size? Determine the flux through the cube in this new situation.

Since the flux through a closed surface that encloses the charge is a constant proportional to the amount of charge enclosed, the increasing of the surface (cube of three times the size) does not change the magnitude of the flux through this new cube. Using Gauss's law it remains the same value because the enclosed charge (+*Q*) remains the same. Therefore, the right side of the Gauss's law expression remains the same.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

$$\therefore \phi = \frac{Q}{\varepsilon_0}$$

The majority of the advanced calculus ideas involved in Gauss's law questions are conceptual in nature. These questions usually have significance when the charge distribution involved in the situation has a spherical symmetry, cylindrical symmetry, or a planar symmetry. If those charge symmetries are involved, then the electric field at the Gaussian surface will be constant. This is significant because it essentially means no integration is necessary for a constant function and the integration basically becomes

$$\phi = \oint \vec{E} \cdot d\vec{A} = E \oint dA = (E)(A)$$

or the product of the electric field through the enclosed area.

Electrostatics: Electric Potential

Definition of Electric Potential

Electric potential is a powerful concept and has many useful relationships that connect electrostatic properties and quantities. We can define the electric potential difference in terms of potential energy. A charge that exists in an external electric field creates a system that can have electric potential energy. The position of the charge in the field will determine the value of electric potential energy of this system. This difference in energy values gives rise to a useful electrostatic property called electric potential (V).

Stated in another way, the change in electric potential energy in moving a charge from point A to point B in an electric field divided by the value of the charge being moved through this difference is called the *electric potential difference*, or ΔV. This quantity is defined as

$$\Delta V = \frac{\Delta U_E}{q}$$

The AP Physics C equation sheet expresses the definition in a slightly different way, although it is mathematically equivalent:

$$\Delta U_E = q\Delta V$$

Example

A charge of $q^{'} = 2x10^{-6}C$ is moved through an electric field by a known outside force. This outside force will change the electrical potential energy of the charge-field system by a known value of $\Delta U_E = +0.2J$. Determine the change in electric potential of the charge.

$$\Delta V = \frac{\Delta U_E}{q^{'}} = \frac{+0.2J}{2x10^{-6}C} = 1x10^5 \frac{J}{C}$$

The units of electric potential are defined as

$$\frac{J}{C} = Volt$$

So the value in the above example could be stated as 100,000 volts.

Definition of Electric Potential Due to a Point Charge

A single point charge creates an electric field in space around the charge. The electric potential at various positions from the charge can be computed using the definition of electric potential due to a point charge. In all cases with single point charges, a value of zero potential is to be considered at an infinite distance from the charge. With this as a reference point the difference in potential at some point, r, is always measured with respect to moving from a position of an infinite distance from the charge to some distance, r, from the charge.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Computing the electric potential due to a single positive point charge of magnitude $q^{'} = 1.0x10^{-9}C$ (or 1.0 nC) for the position of $r = 1\ m$ from the charge would look like this:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = \left(9x10^9\frac{Nm^2}{C^2}\right)\frac{1x10^{-9}C}{1m} = 9\ V$$

So at a distance of 1 m away from a 1 nC charge the electric potential has a value of 9 volts.

Definition of Electric Potential Due to a Collection of Point Charges

There are many instances when multiple point charges are to be considered acting in a particular region of space, as shown in figure 9.1.

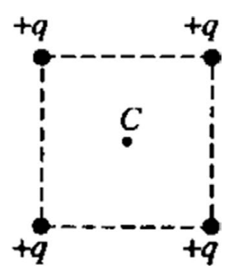


Figure 9.1: Four Point Charges Arranged in a Square

Example

Given the four point charges of $+q$ arranged in figure 9.1, determine the electric potential at the center of the square.

The electric potential due to a collection of charges is simply the sum of the electric potentials due to each individual charge (q_i) and each position to each charge (r_i). The definition is

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}$$

Assume the distance of each side of the square is d = 1.0 m. The magnitude of each charge is $q = 1.0\times10^{-9}C$.

First, we need to determine the distance from each charge to the center of the square. The diagonal of the square would be $\sqrt{2}\,m$. Since the distance from each charge to the center is half the diagonal distance, then the distance of each charge to the center of the square is

$$r = \frac{\sqrt{2}}{2} = 0.71m$$

Now we'll compute the electric potential:

$$V_{center} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left(\frac{1\times10^{-9}}{0.71m}+\frac{1\times10^{-9}}{0.71m}+\frac{1\times10^{-9}}{0.71m}+\frac{1\times10^{-9}}{0.71m}\right)$$
$$V_{center} = 51V$$

General Definition of the Electric Potential Energy of Two Point Charges

If we know the potential at some position r due to a single point charge, then the amount of electrical potential energy in the system of two point charges is simply

$$U_E = \Delta V \cdot q_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

where q_1 is the source charge and q_2 is the charge experiencing the electric field of q_1. The distance r is the distance between the charges, as the potential is measured at the location of charge q_2.

General Definition of Potential Difference

The general definition of electric potential difference is

$$\Delta V = \frac{\Delta U_E}{q} = \frac{W_{A\to B}}{q}$$

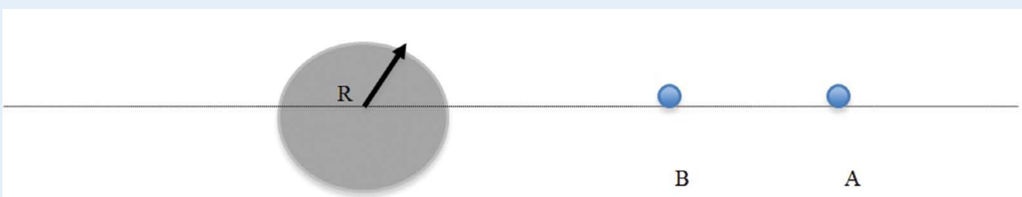
Using the general definition of a conservative force and the calculus definition of work, this relationship can be transformed into a general calculus relationship that relates the difference in potential between two points in a field to a general integral relationship:

$$\Delta V = V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{r}$$

where *dr* means that the difference in the two points can be along the radial direction (which is often the case in the AP Physics C course, as many of the charge distributions create radial fields).

Example

A metal sphere of radius R has a charge Q on the surface of the sphere. Using the definition of potential difference, determine the potential difference between two arbitrary points in space as shown here:



In this example, point A is located a distance of $4R$ from the center of the metal sphere and point B is located a distance of $3R$ from the center of the sphere. The electric field outside of a charged metal sphere is defined as

$$E = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r}$$

and executing the integral definition and the dot product gives

$$\Delta V = V_b - V_a = -\int_{4R}^{3R}\left[\frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}\right]dr = -\left[-\frac{1}{4\pi\varepsilon_o}\frac{Q}{r}\right]_{4R}^{3R}$$

$$\Delta V = V_b - V_a = \left[-\frac{1}{4\pi\varepsilon_o}\frac{Q}{r}\right]_{4R}^{3R} = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{3R} - \frac{1}{4R}\right]$$

$$\Delta V = V_b - V_a = \frac{Q}{4\pi\varepsilon_o}\frac{1}{12R} = \frac{Q}{48\pi\varepsilon_o R}$$

This potential difference shows that point B is at a higher potential than point A. This result is a positive potential difference value. Note: There are some subtle details in this calculation that are not completely shown here. Please use your textbook and other resources to completely learn all of the details of the mathematics in this solution.

Differential Relationship

Another way to write the relationship between the potential and the electric field is to use a differential relationship, which looks like this:

$E_r = -\frac{dV}{dr}$ in radial dimensions, or $E_x = -\frac{dV}{dx}$ in Cartesian dimensions.

In order to use this relationship, the potential (V) would have to be defined as a function of position.

Example

Given a potential function that varies with the x-direction in the following way:

$$V(x) = A - Bx$$

where $A = 30\ Volts$, $B = 3\frac{V}{m}$, and x is measured in meters, determine the difference in potential between a point on the x-axis at 10 m and the point $x = 0$.

$$V(x) = 30 - 3x$$
$$V(10) = 30 - 3(10) = 0$$
$$V(0) = 30 - 3(0) = 30$$
$$\Delta V = V(10) - V(0) = 0 - 30 = -30\,volts$$

Determine the electric field in the x-direction.

$$E_x = -\frac{dV}{dx} = -\frac{d\left[30 - 3x\right]}{dx} = 3$$

The units of the electric field would be $\frac{V}{m}$ (which is equivalent to N/C). The direction of the electric field would be in the positive direction, which is indicated by the positive value for the 3 V/m.

Capacitance

Definition of Capacitance

The most basic model of a **capacitor** is the parallel plate capacitor. A parallel plate consists of two large metal conductive plates that are separated by a very small distance. Equal and opposite amounts of charge are placed on the plates via some electrical process. Sometimes a dielectric is placed in between the plates to allow for more charge to be stored on the plates. An electric field and potential difference between the plates is developed as more and more charge is placed on the plates. The capacitor allows for this charge and energy to be stored and used at a later time.

Capacitance is defined as the ratio of two physical quantities:

$$C = \frac{Q}{\Delta V}$$

The units of capacitance are Farads (F).

$$Farad = \frac{Coulomb}{Volt} = \frac{Coulomb^2}{Joule}$$

Example

A standard parallel plate capacitor has a total stored charge of $Q = 1x10^{-6}C$ on its plates. The plates have a measured potential difference of 10.0 volts. Determine the capacitance of this capacitor.

An interesting point to remember is that the net charge on a capacitor is always zero. This is because the two plates have equal but opposite charges. The amount of charge used in the definition of capacitance is never zero, but is the value of the charge on one plate.

$$C = \frac{1x10^{-6}C}{10V} = 1x10^{-7}F$$

This can also be stated as 100 nF or 0.1 μF.

If you need more information, the following tutorial can help to further explain this concept:

Khan Academy: Electric potential at a point in space

A Definition of a Parallel Plate Capacitor

It turns out that the ratio of charge to potential difference is also equivalent to the geometrical properties of a capacitor, the dielectric properties of a capacitor, and the permittivity of free space. This definition shows precisely that the capacitance of the capacitor depends solely on the area of the conductive plates, the distance between the plates, and the permittivity of the dielectric medium. This definition is

$$C = \frac{\kappa\varepsilon_0 A}{d}$$

where A is the area of the plates and d is the distance between the plates.

The constant ε_0 is defined as the permittivity constant and κ is defined as the dielectric constant, a dimensionless constant that gives a description of the polarizability of the atoms in the material.

Example

Using the example above of the 100 nF capacitor, let's assume the capacitor has a dielectric in between the plates with $\kappa = 4$ and a distance of separation of $d = 1.0x10^{-4}m$. Determine the size of the area of the plates for a model of a parallel plate capacitor.

$$C = \frac{\kappa\varepsilon_0 A}{d}$$

$$\Rightarrow A = \frac{Cd}{\kappa\varepsilon_0} = \frac{\left(100x10^{-9}\frac{C^2}{J}\right)\left(1x10^{-4}m\right)}{(4)\left(8.85x10^{-12}\frac{C^2}{Nm^2}\right)} = 0.28\ m^2$$

This is a fairly large plate size — about the size of large baking tray. A capacitor of that size could not fit into most electronic components. Most capacitors are tiny and have smaller plates squeezed very close together by using a large dielectric. The manufacturers also typically roll the capacitors into cylindrical shapes to minimize volume.

Energy Stored in a Capacitor

The capacitor is an electrical device that stores both charge and energy. The amount of charge stored is implicitly defined in the ratio definition of capacitance. Here is the definition of energy stored by a capacitor:

$$U_E = \frac{1}{2}C(\Delta V)^2$$

By using the definition of capacitance and some algebra, one can show this definition in two other equivalent expressions:

$$U_E = \frac{Q^2}{2C} = \frac{1}{2}Q\Delta V$$

Example

Using the 100 nF capacitor from the previous examples, determine the energy stored when a 10 volt potential difference is applied to the capacitor.

$$U_E = \frac{1}{2}C(\Delta V)^2 = (0.5)\left(1x10^{-7}\,F\right)(10V)^2 = 5x10^{-6}\,J$$

or $5\mu J$ of stored energy.

Current, Resistance, and Circuits

Current

Current is the rate of charge moving past a given point in a conductor. The steady state definition of current is

$$I = \frac{Q}{\Delta t}$$

The (SI) unit of current is ampere (A):

$$\frac{C}{s} = A$$

Example

Determine the current in SI units, if a mole of electrons passes by a reference point in one hour.

$$1mole = 6.023x10^{23}\,electrons$$

$$Q = \left(6.023x10^{23}\,electrons\right)\left(1.6x10^{-19}\,\frac{C}{e}\right) = 9.6x10^{4}\,C$$

Giving a current of

$$I = \frac{Q}{\Delta t} = \frac{9.6x10^4 C}{3600s} = 26.8A$$

This is a very large current! It demonstrates that a mole of charge is a very large quantity of charge.

Microscopic Definition of Current

The current can also be defined by a conductor's properties (size, shape, charge density). This definition is sometimes called the microscopic definition of current. The definition is

$$I = NeAv_d$$

where N is the electron density (charge/volume), A is the cross sectional area of the conductor, v_d is the drift velocity of electrons, and e is the charge on the electron.

Example

Using our previous example with current, determine the drift velocity in a conductor that has a cross sectional area of $A = 0.01\ m^2$ and a length of 1 meter.

The current value obtained in the last example was approximately 27 amperes. The number of charge carriers in the example was one mole of electrons. Using the microscopic definition of current, the drift velocity of one mobile charge carrier can be determined.

$$N = electron\ density = \frac{6.023x10^{23} e}{(0.01m^2)(1m)} = 6.023x10^{25} m^{-3}$$

$$I = NeAv_d \Rightarrow v_d = \frac{I}{NeA} = \frac{27A}{\left(6.023x10^{25}\right)\left(1.6x10^{-19} C\right)\left(0.01\ \mathrm{m}^2\right)} = 2.8x10^{-4} \frac{m}{s}$$

The drift velocity is a surprisingly small value. This is typically the case in most conductors. The drift velocity is on the order of fractions of cm/s. Large values of current are created not by fast moving electrons, but by the transfer of a large amount of charge (moles or greater).

Calculus Application with the Definition of Current

If the current is a transient current (changing with time), then the relationship can be written with differentials like this:

$$I = \frac{dQ}{dt}$$

Using this expression allows one to compute the amount of charge that has passed through a conductor in a given unit of time by using calculus:

$$Q = \int I\,dt$$

Example

Given a transient current in a circuit of $I(t) = I_0 e^{-kt}$, determine the amount of charge that is moved through the circuit in one minute of time.

$$Q = \int_0^{60} I_0 e^{-kt} dt$$

$$Q = -\frac{I_0}{k} \cdot e^{-kt}\Big|_0^{60} = -\frac{I_0}{k}\left(e^{-60k} - e^0\right) = \frac{I_0}{k}\left(1 - e^{-60k}\right)$$

To give this computation some practical measurements, let's make the initial current value $I_0 = 2x10^{-3} A$ or 2mA, and the k value $k = 0.002\ s^{-1}$. Now we can get a value for the charge that moved through the circuit in one minute of time.

$$Q = \frac{I_0}{k}(1 - e^{-60k}) = \frac{2x10^{-3} A}{0.002 s^{-1}}\left(1 - e^{-(60 \cdot 0.002)}\right) = 1(0.113) = 0.113\ C$$

Resistance and Circuits

A property of a conductor that is significant in determining the flow of charge in a conductor is the property of **resistance**. Resistance in a conductor is defined in the following way:

$$R = \frac{\rho \ell}{A}$$

where ρ is resistivity measured in $\Omega \cdot m$, ℓ is the length of the conductor measured in meters, and A is the cross sectional area the conductor measured in m^2. Resistivity is an inverse relationship of conductivity of a conductor and is related to the electron density of the conductor.

Adding Resistors

Now we will show the rules for adding resistors in the two particular ways that circuit devices can be arranged. Resistors can be arranged in series (figure 9.2), in parallel (figure 9.3), or in some advanced network or combination of these two arrangements.

Series Arrangement:

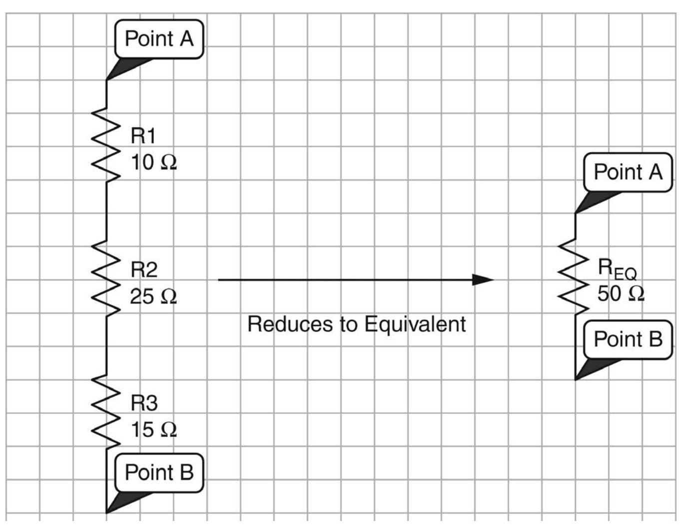


Figure 9.2: Resistors Arranged in Series

Example

Given the three resistors shown in figure 9.2 (10Ω, 25Ω, and 15 Ω), show how the equivalent circuit behaves.

The rule for resistors in series is very simple: $R_E = \sum_n R_n$, or simply the algebraic sum of the resistors placed in series. So for our example it becomes:

$$R_E = 10\Omega + 25\Omega + 15\Omega = 50\Omega$$

In other words, the three resistors have the same equivalent property of one resistor of 50 ohms. Essentially a row of resistors is equivalent to having one equivalent resistor of a longer length.

Parallel Arrangement:

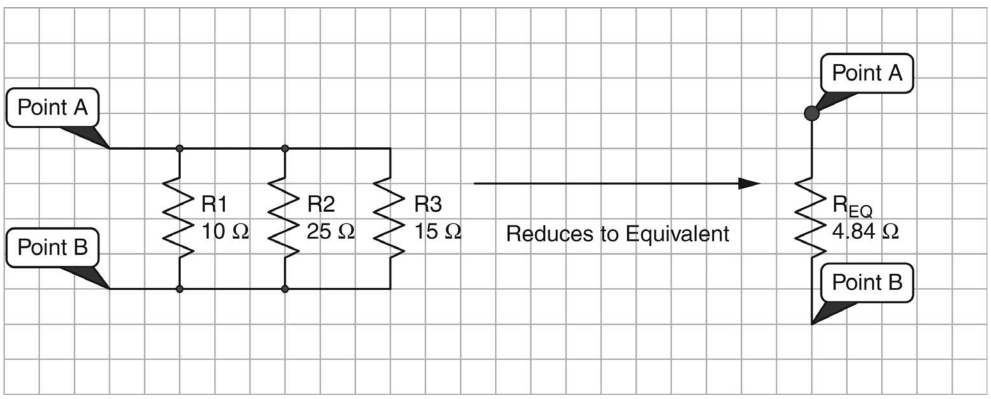


Figure 9.3: Resistors Arranged in Parallel

Example

Given the three resistors shown in figure 9.3, show the equivalent resistance of this arrangement.

Using the same three resistor values as the series arrangement, we will apply the rule for parallel resistors:

$$\frac{1}{R_E} = \sum_n \frac{1}{R_n}$$

So, applying this rule to our example gives:

$$\frac{1}{R_E} = \frac{1}{10} + \frac{1}{25} + \frac{1}{15} = \frac{15}{150} + \frac{6}{150} + \frac{10}{150} = \frac{31}{150}$$

$$\frac{1}{R_E} = \frac{31}{150}$$

$$R_E = \frac{150}{31} = 4.84\Omega$$

Notice that in the parallel arrangement the equivalent resistance is less than the smallest resistor in the arrangement. The physics behind why the resistors behave this way are beyond the scope of this chapter, but you should review your textbook to explore this issue in more depth.

If you need more information, the following tutorials can help to further explain these concepts:

Khan Academy: Resistors in series

Khan Academy: Resistors in parallel

Ohm's Law

Ohm's law is the fundamental law in circuit behavior. It relates the three fundamental physical characteristics of circuits: potential difference, current, and resistance. Ohm's law is valid at every point in a circuit, across every branch in a circuit, and for the entire equivalent circuit. Ohm's law is typically written as

$$I = \frac{\Delta V}{R}$$

showing that current in a conductor or pathway in a circuit is proportional to the potential difference across that path and inversely proportional to the resistance of the conductive path.

This also means that the unit for resistance (Ohm) is equivalent to

$$Ohm = \frac{Volt}{A} = \frac{Js}{C^2}$$

Microscopic Definition of Ohm's Law

If the microscopic definition of current is combined together with the Ohm's law relationship, another relationship can be obtained that relates the electric field that drives the mobile charges in the conductor to the rate of the charge passing through the conductor. This relationship is

$$\vec{E} = \rho \vec{J}$$

$\vec{J}$ is defined as current density (current/area) and is a vector in the same direction as the conventional current definition.

Example

Determine the value of the electric field within a conductor that drives electrons at a current of 1.0 ampere in a 14 gauge copper wire.

The copper wire has a resistivity of $\rho = 1.7 \times 10^{-8}\,\Omega \cdot m$ (this is a property of copper and can be found in handbooks, tables, or textbooks). A 14-gauge wire has a cross sectional area of $A = 2.1x10^{-6}m^2$ (this value can also be looked up in electrical handbooks).

The current density of this particular wire is

$$\vec{J} = \frac{I}{A} = \frac{1A}{2.1x10^{-6}m^2} = 4.76x10^5 \frac{A}{m^2}$$

Then use the microscopic definition of Ohm's law to give the electric field:

$$\vec{E} = \rho\vec{J} = (1.7x10^{-8}\Omega m)(4.76x10^5 \frac{A}{m^2}) = 8.1x10^{-3} \frac{N}{C}$$

This example shows that it does not take a very large electric field within a conductor to produce a very large current (in a good conducting medium).

Companion to Ohm's Law: The Power Relationship

It is often useful to discuss the amount of energy transferred in an electrical device or circuit. The companion relationship that pairs with Ohm's law is the power relationship, which is

$$P = I\Delta V$$

where P is power measured in watts, ΔV is potential difference measured in volts, and I is current measured in amperes.

This relationship gives the power developed by an electrical device in the *SI* unit of watts. So, if the two electrical properties (resistance and potential difference) are known or given then, the other two properties (current and power) can be determined by using Ohm's law and this companion relationship for power.

Example

A battery of 12 volts is attached to an electrical device that has a resistance of 6 ohms. Determine the amount of energy transferred by the battery in one minute of operation.

First, determine the current developed in the circuit.

$$I = \frac{\Delta V}{R} = \frac{12V}{6\Omega} = 2A$$

Now, using the current value, the power developed by the circuit can be determined using the power relationship.

$$P = I\Delta V = (2A)(12V) = 24\ W$$

Since we now have the power developed in the circuit, we can use the definition of power to determine the amount of energy transferred:

$$P = \frac{\Delta E}{\Delta t} \Rightarrow \Delta E = P\Delta t = \left(24\frac{J}{s}\right)(60s) = 1440\ J$$

The battery transfers a total of 1440 joules to the electrical device in one minute of operation.

Adding Capacitors in Circuits

Capacitors are circuit devices that have specific properties and behave in certain ways in a circuit. One useful property of these circuit devices is that they add in specific ways when they are placed in certain orientations within a circuit. These devices can be arranged in series (figure 9.4) or in parallel (figure 9.5). They can also be arranged in complex networks or combinations of these two types of arrangements. The basic properties of this addition are shown in the following examples.

Series Arrangement

Example

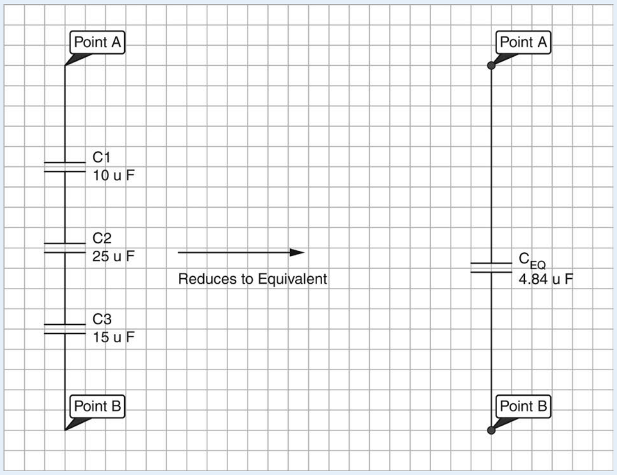


Figure 9.4: Capacitors Arranged in Series

Given three capacitors arranged in series as shown in figure 9.4, determine the equivalent capacitance of the arrangement.

The circuit shown above has three capacitors (10 μF, 25 μF, and 15 μF) arranged in series. The effect of having three capacitors in series is that it will behave as if it were one capacitor with an effective or equivalent value. The capacitors in series add in the following way:

$$\frac{1}{C_E} = \sum_n \frac{1}{C_n} = \frac{1}{10} + \frac{1}{25} + \frac{1}{15} = \frac{15}{150} + \frac{6}{150} + \frac{10}{150} = \frac{31}{150}$$

$$\frac{1}{C_E} = \frac{31}{150}$$

$$C_E = \frac{150}{31} = 4.84\mu F$$

So the three capacitors in the circuit shown above behave as if it were one capacitor with a value of 4.84 μF.

Note that the equivalent capacitance of a set of capacitors in series is always less than the smallest capacitance in the series.

If you need more information, the following tutorial can help to further explain this concept:

Khan Academy: Capacitors in series

Parallel Arrangement

Example

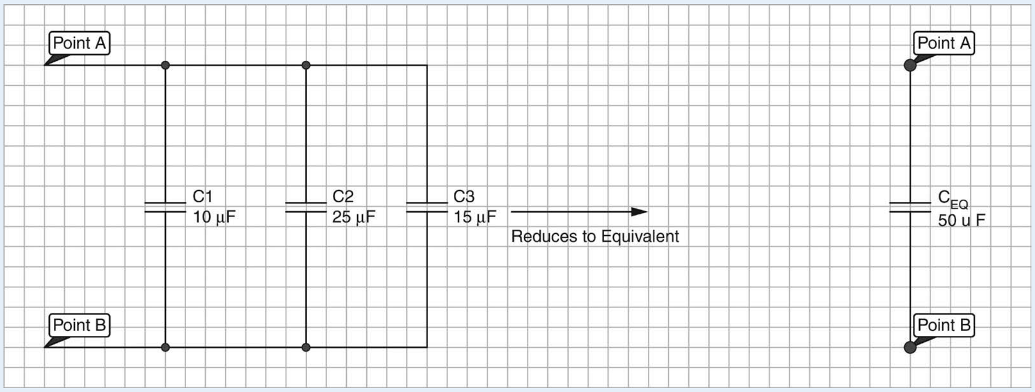


Figure 9.5: Capacitors in Parallel Arrangement

Figure 9.5 shows three capacitors placed in parallel with each other. Determine the equivalent capacitance of the three capacitors. To draw a comparison between the two methods (parallel and series), we will use the same capacitor values as the series example.

The rule for adding capacitors in parallel is

$$C_E = \sum_n C_n$$

or simply adding the capacitors. So the equivalent capacitance of the above circuit is

$$C_E = 10\mu F + 25\mu F + 15\mu F = 50\mu F$$

This value is more than 10 times the value of the equivalent capacitance of the same three capacitors arranged in series. The reason that the parallel arrangement of capacitors results in a much larger equivalent capacitance is essentially that the three different areas of each of the capacitors are all available for charge to fill up. So, the three capacitors are essentially creating one large capacitance of an equivalent larger area.

If you need more information, the following tutorial can help to further explain this concept:

Khan Academy: Capacitors in parallel

Magnetism

Moving Charge in a Magnetic Field

It was discovered in the 1800s that moving charge is the fundamental cause of a magnetic field. If a moving charge is moving through a magnetic field, the two fields will interact with each other and produce what is called a *magnetic force*. The two fields are the magnetic field produced by the moving charge itself and the field that it is moving through. This idea is used in the area of particle physics to determine particle characteristics (charge, mass, etc.).

The force of interaction of these two fields always creates a force that is perpendicular to the field direction and to the velocity vector. This is described by the cross product in the definition

$$\vec{F}_M = q(\vec{v} \times \vec{B})$$

where B is the external magnetic field measured in Teslas, q is the magnitude of moving charge, and v is the velocity of the charge.

Example

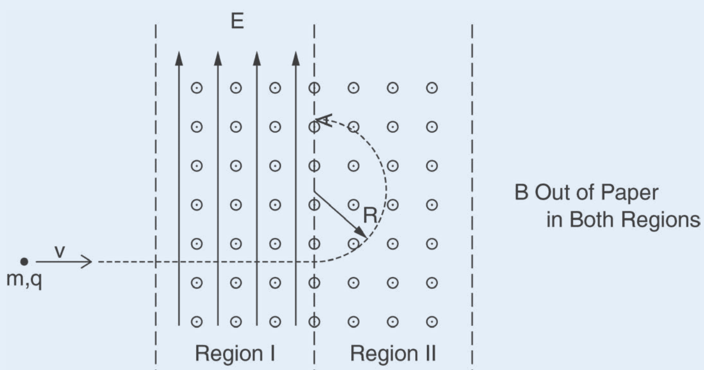


Figure 9.6: A Charged Particle Moving Through a Magnetic Field

Determine the magnitude and direction of the magnetic force on the moving charged particle shown in figure 9.6. The charged particle moves through two regions. In region II, the charged particle is moving through only B field.

The moving charge is an *electron* (-e) and has a speed of $\left|\vec{v}\right| = 1x10^5 \frac{m}{s}$. The magnitude of the external magnetic field is $B = 2.0\ T$.

To determine the force direction we will apply the definition of magnetic force for a moving charge. Note that the direction is determined by the cross product and the velocity vector is defined to be the velocity vector of a positive moving charge. This is a very important detail.

The charge is shown in figure 9.6 to take a circular path (the nature of the magnetic force produces a circular trajectory). The initial magnetic force direction is vertically upward (+y direction). This comes from the cross product of a velocity vector to the left (-x direction) and the magnetic field shown out of the page (+z direction). Remember the rule for a negative charge is that the velocity vector is the opposite of the actual velocity (-x direction): this is to maintain the right-handed system. The cross product of the two vectors gives a force vector that is directed vertically upward as shown in figure 9.6.

The magnitude of the force is

$$\vec{F}_M = q\left(\vec{v}\times\vec{B}\right)$$

$$\left|\vec{F}_M\right| = q\left(\left|\vec{v}\times\vec{B}\right|\right) = qvB\sin\theta$$

$$\left|\vec{F}_M\right| = qvB\sin\theta = \left(1.6x10^{-19}C\right)\left(1x10^{5}\frac{m}{s}\right)\left(2.0\ T\right)\sin 90^{\circ} = 3.2x10^{-14}N$$

This is a small magnitude indeed, but certainly large enough to deflect the tiny electron $(m_e = 9.1\times10^{-31}kg)$.

If you need more information, the following tutorial can help to further explain this concept:

Khan Academy: Magnetic force on a charge

Current-Carrying Conductor (Wire) in a Magnetic Field

Figure 9.7 shows a wire with a current (I), stationary in an external magnetic field (B). The current is shown in the positive y-direction and the magnetic field is shown in the positive x-direction. There will be a magnetic force on the wire due to the two fields interacting (the field of the moving charge in the wire and the external field).

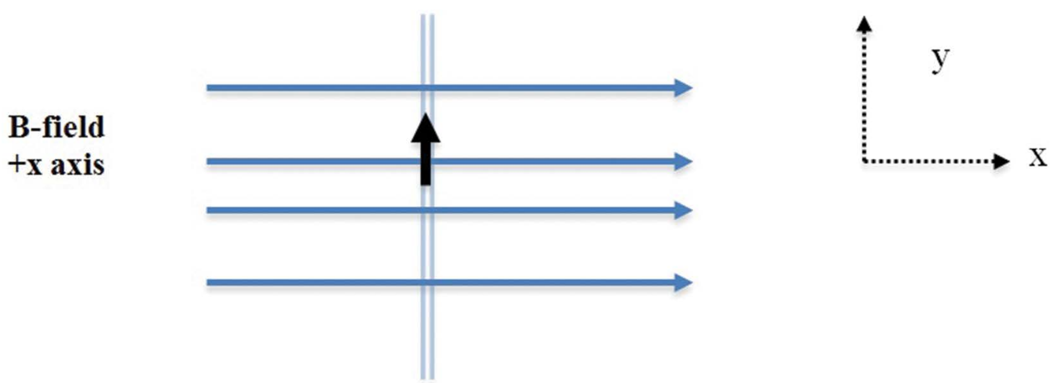


Figure 9.7: Wire with Current in a Magnetic Field

The AP Physics C equation sheet represents the magnetic field on a length of current as

$$\vec{F}_M = I\int d\vec{\ell}\times\vec{B}$$

This can also be expressed without the integration in the following way:

$$\vec{F}_M = I\left(\vec{\ell} \times \vec{B}\right)$$

$$\left|\vec{F}_M\right| = I\left|\vec{\ell}\right|\left|\vec{B}\right|\sin\theta$$

where θ is the angle between the current direction (or wire length in the current direction) and the magnetic field vector, and ℓ is the length of the wire in the field (or interacting with the field).

Example

Using figure 9.7 and the following parameters, determine the magnetic force on the wire.

I = 2.0 amperes, length of wire in field $\ell = 0.5m$, and magnitude of magnetic field B = 2.0 T.

$$\left|\vec{F}_M\right| = \mathrm{I}\ell\mathrm{B}\sin\theta = (2A)(0.5m)(2T)\sin 90^\circ = 2.0\ N$$

The angle between the conductor and the magnetic field is 90 degrees. These two vectors are mutually perpendicular to each other. Applying the cross product correctly determines that the force vector will be directed in the -z direction (or into the page!).

Note 1: The actual AP Physics C equation sheet shows the following for the magnetic force due to a wire:

$$F_M = I\int d\vec{l} \times \vec{B}$$

This is precisely correct, but this definition is rarely used in this form. The wires interacting with fields in the AP Physics C course will be straight. This would eliminate the need to integrate over a differential length of the wire. Therefore, the most useful definition of the magnetic force on a wire is

$$F_M = I(\vec{\ell} \times \vec{B})$$

$$\left|\vec{F_M}\right| = \left|\vec{B}\right| I\ell\sin\theta$$

Note 2: You should review the right-hand rules to help with quickly determining force directions due to magnetic interactions. The cross product is the actual definition, but the right-hand rules create an easy memory device to make the determinations quickly and accurately.

If you need more information, the following tutorials can help to further explain these concepts:

Khan Academy: Magnetic force between two currents going in the same direction

Khan Academy: Magnetic force between two currents going in opposite directions

Biot–Savart Law

The Biot–Savart law is very similar to the law used in electrostatics to compute any electric field due to a continuous line charge of charge density:

$$E = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}$$

Using this allows the computation of an electric field due to a wire of continuous charge or a ring of continuous charge.

The Biot-Savart law allows for the computation of magnetic fields due to continuous current distributions (like current in a wire or ring). The Biot–Savart law in differential form is

$$dB = \frac{\mu_0}{4\pi}\frac{I(d\vec{\ell}\times\hat{r})}{r^2}$$

Notice how it looks very similar in structure to the companion law in electric fields.

Example

Using the Biot–Savart law, determine the magnetic field at the center of the ring of radius R and current I.

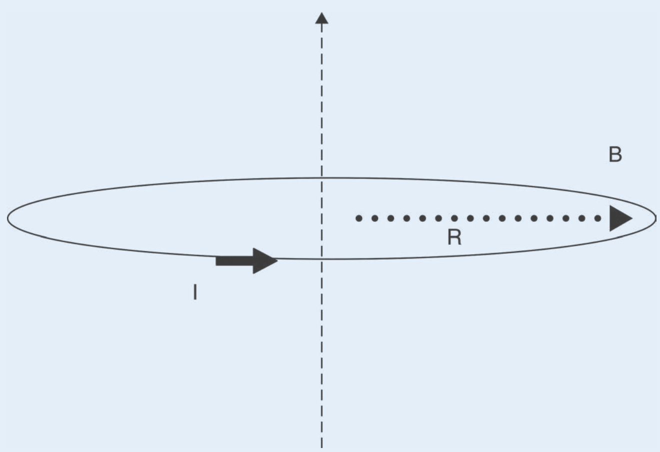


Figure 9.8: A Ring of Radius R, with Current I in CCW Direction Indicated on Wire

The most difficult part of the Biot–Savart law is the cross product of the $d\vec{\ell}$ and $\hat{r}$. The $d\vec{\ell}$ vector is in the direction tangent to the current (tangent to circle). The $\hat{r}$ vector is always radial and perpendicular to the $d\vec{\ell}$ vector. This means that the cross product of the two is a vector that is in the y-direction (B-direction!). The direction of the cross product of the numerator is the direction of the B-field. Note that since the angle between $d\vec{\ell}$ and $\hat{r}$ is always 90 degrees the contribution of the sine of 90 degrees is one, which simplifies the expression.

$$dB = \frac{\mu_0}{4\pi}\frac{I(d\vec{\ell}\times\hat{r})}{r^2}$$

$$B = \int \frac{\mu_0}{4\pi}\frac{I(d\vec{\ell}\times\hat{r})}{r^2}$$

Now everything is a constant in the above expression except $d\vec{\ell}$. The position vector r will become the radius of the ring (R). This is because we are computing the magnetic field for the center of the ring, which is precisely a distance of R from the ring. Now let's rewrite the expression above and simplify the result:

$$B = \int \frac{\mu_0}{4\pi} \frac{I(d\vec{\ell} \times \hat{r})}{r^2}$$

$$B_{center} = \frac{\mu_0}{4\pi} \int \frac{I(d\vec{\ell} \times \hat{r})}{r^2} = \frac{\mu_0}{4\pi} \int \frac{I(d\ell)}{r^2}$$

$$B_{center} = \frac{\mu_0}{4\pi} \frac{I}{R^2} \int d\ell$$

Now we are down to one last mathematical step: the integral of $d\ell$. The result of this integral is simply the full circumference of the ring. You have integrated the infinitesimal lengths of the ring around the entire ring, so the result of this is the full length of the ring $2\pi R$.

$$B_{center} = \frac{\mu_0}{4\pi} \frac{I}{R^2} \int d\ell = \frac{\mu_0}{4\pi} \frac{I}{R^2} \cdot (2\pi R) = \frac{\mu_0 I}{2R}$$

This is the known expression for the magnitude of the magnetic field of the ring of current at the center of the ring.

Ampere's Law – An Analog to Gauss's Law in Magnetism

There is also a fundamental law in magnetism that relates the current of a conductor (or multiple conductors) to the magnetic field that the current creates around the conductor(s). This law is called Ampere's law:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

where μ_0 is the magnetic permeability constant and I is the enclosed current.

The statement above is called a line integral. The line integral means that the product of B and some differential length element $d\ell$ are computed around a complete pathway. Sometimes the pathway is a simply a line, but in the case of magnetic fields symmetry our path lengths will match the symmetry of the magnetic field of a wire — a circular path.

Conceptually this means that the magnetic field that encircles a wire or collection of wires carrying current is proportional to the amount of current contained (enclosed) by an imaginary closed loop that contains the current. This imaginary closed loop is called an Amperian loop.

The law is similar to Gauss's law in that it takes a very complex calculus idea and uses symmetry arguments to make the calculations very simple. The point of using Ampere's law (and Gauss's law) is to avoid doing a complex integration!

Using Ampere's Law to Derive the Magnetic Field Due to a Wire

Many physics students have memorized the magnetic field due to a single wire carrying a current. The field expression is

$$B = \frac{\mu_0 I}{2\pi r}$$

We can use Ampere's law to come up with this expression.

Example

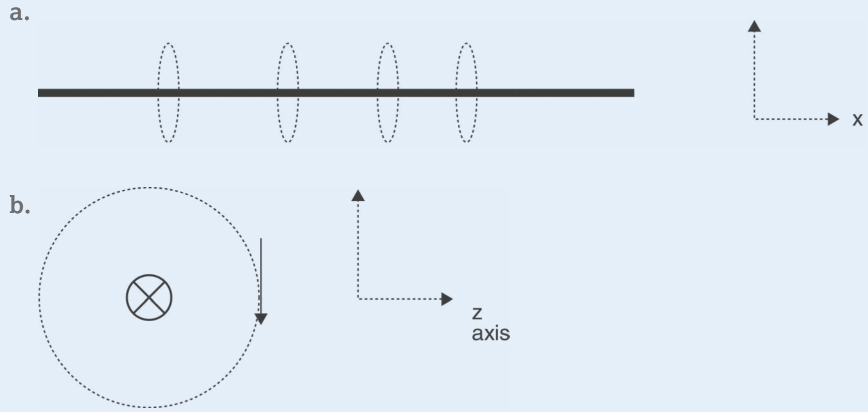


Figure 9.9: Two Views of Magnetic Field Around a Wire

Figure 9.9a shows a wire with a current (I) directed in the $+x$ direction. A magnetic field encircles the wire. Figure 9.9b shows the view from the end of the wire. Current is shown by vector symbol x (into the page) and the dashed circle represents the magnetic field around the wire. A vector is drawn tangent to the circular field line: this vector represents the B-field at this point. The Amperian loop is a simple closed circle that matches the circular nature of the magnetic field. The vector associated with $d\vec{\ell}$ would be an identical vector, tangent to the dashed circular path. The dot product of the two vectors would simply yield $|\vec{B}|\cdot|d\vec{\ell}|$ since both vectors are co-linear. Now apply the law:

$$\oint \vec{B} \bullet d\vec{\ell} = B\oint d\ell = B\ell$$

and in this case the total length of the Amperian loop is the circumference. If we call the distance to the Amperian loop the position vector r, then the circumference of the loop is simply $2\pi r$. The enclosed current by this loop is I. Applying the integration and solving for the magnitude magnetic field B gives:

$$\oint \vec{B} \bullet d\vec{\ell} = \mu_0 I$$

$$B(2\pi r) = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

This example shows the simplicity and fundamental nature of Ampere’s law.

If you need more information, the following tutorial can help to further explain these concepts:

Khan Academy: Magnetic field created by a current carrying wire

Magnetic Field of an Ideal Solenoid

The solenoid is what most people imagine when they think of the classic coil magnet. A standard solenoid (figure 9.10) is a coil of many tightly wound turns around a solid core.

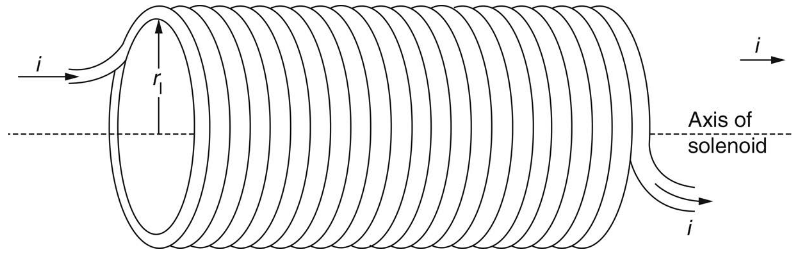


Figure 9.10: Example of a Standard Solenoid

The solenoid is a device that is often used in physics experiments to create a fairly constant magnetic field. When current is present in the solenoid, the magnetic field can be fairly large and is constant along the axis of the solenoid (except at the two ends of the solenoid). Typical laboratory solenoids are about 20–30 cm in length and have about 1000 turns. These solenoids typically produce magnetic fields in the 10^{-3} range. The definition for the magnetic field of the solenoid is

$$B_s = \mu_0 n I$$

The symbols in the expression are defined as follows: μ_0 is the magnetic permeability constant, n is the turn density or number of turns per length of solenoid, and I is the magnitude of current in the solenoid.

Example

Determine the magnetic field of a solenoid of length 25 cm, with 1000 turns and a current of 2.0 amperes.

The turn density of this solenoid is $n = \frac{1000\ turns}{0.25\ m} = 4000\frac{turns}{m}$.

Then

$$B = \mu_0 n I = \left(4\pi x 10^{-7}\frac{T\cdot m}{A}\right)\left(4000\ m^{-1}\right)\left(2\ A\right) = 0.01\ T$$

or about 10 mT.

The direction of the magnetic field in the above example is directed along the axis of the solenoid to the right in the plane of the paper. It would be a constant value in the entire space inside the solenoid. To determine the direction, one would wrap their right hand in the sense of the current curl and the thumb would point in the field's direction.

If you need more information, the following tutorial can help to further explain this concept:

Khan Academy: Using the right-hand rule

Induction and Inductance

Magnetic Flux

The **magnetic flux** is defined qualitatively as the magnitude of the magnetic field that permeates space through a particular defined area. The precise mathematical definition is

$$\phi_B = \int \vec{B} \cdot d\vec{A}$$

Magnetic flux is very useful in the magnetism part of the course and is a significant physical feature of one of the most fundamental laws of electromagnetism — Faraday's law (defined in the next section).

There tends to be more calculus involved in working with magnetic flux. One reason is that some of the more useful (and often-used) magnetic fields are fields that can vary over the surface of the area in question. Another reason is that one of the fundamental principles of magnetism involves a changing flux over time, which leads to more mathematical understanding and rigor with the use of magnetic flux. An example of a flux calculation of a field that changes in value over the area is beyond the scope of this chapter, but you should certainly work with this idea using your textbook.

Faraday's Law and Lenz's Law

The fundamental nature of electromagnetism is that a changing magnetic field creates a changing electric field and each gives rise to the other field, creating an electromagnetic (EM) wave if it is changing fast enough. Faraday's law is expressed mathematically as

$$\varepsilon = -\frac{d\phi_B}{dt}$$

If a magnetic field permeates a conductive loop (area) and this field begins to change (i.e., increase, decrease, oscillate, or even simply turn off or on) then a current is created in that conductive loop that will oppose the changing field that has created it. This opposing part of the law also has a name: this law is called Lenz's law. Both of these laws are extremely important in this part of the course.

Example

Determine the induced current that exists in the current loop shown in figure 9.11.

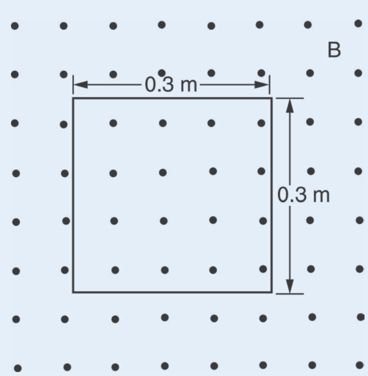


Figure 9.11: B Field Permeating Conductive Square Loop

The magnetic field is changing and has the following relationship with time:

$$B = B_0 - kt$$

where k is a constant measured in $\frac{T}{s}$. The initial field B_0 is initially directed out of the page as the vector symbols show in the figure. The field decreases until it reaches a value of zero. The conductive loop has a resistance of R.

To determine the induced current we need to determine the flux through the square loop.

$$\varphi_B = \int \vec{B} \cdot d\vec{A} = B \int dA = B \cdot A$$

The magnetic field is changing with time, but not changing over the area. It has a constant value (at any one instant in time) over the entire square area. This is why the flux integral reduces down to simply the product of field and area.

$$\varphi_B = B \cdot A = (B_0 - kt)(0.3m)(0.3m) = 0.09(B_0 - kt)$$

Now let's take the derivative of this expression to obtain the induced **electromotive force**, or emf.

$$\varepsilon = -\frac{d\varphi_B}{dt} = \frac{d}{dt}(0.09(B_0 - kt)) = -(-0.09k) = +0.09k$$

This value is the induced voltage in the loop. We can then determine the induced current in the loop by using Ohm's law.

$$i = \frac{0.09k}{R}$$

The positive value that is associated with this current (and induced voltage) is a current that would be counter clockwise (CCW) directed around the square loop. This is a result of Lenz's law and the fact that a diminishing or decreasing field results in a current that will create a field in the same direction as the changing field. The change is inward on the page (-z direction). Lenz's law states the induced current will oppose the change and opposing the change would be in the opposite direction of the decreasing field, which is back in the same direction as the original direction of the field (at time zero).

If you need more information, the following tutorial can help to further explain these concepts:

Khan Academy: Flux and magnetic flux

Induced EMF in an Inductor

An inductor is an electrical device that stores and transfers energy and is very useful in AC circuits. An inductor creates a back emf or induced emf in the circuit whenever a current is changing within a circuit. The definition for this induced emf is a direct consequence of Faraday's law. The definition is

$$\varepsilon_L = -L\frac{dI}{dt}$$

where L is the inductance measured in henrys and $H = \Omega \cdot s$.

Example

Determine the induced emf in an inductor of $L = 2\ mH$, when the current is changing at a rate of $\frac{di}{dt} = +2.0\frac{A}{s}$.

$$\varepsilon_L = -L\frac{di}{dt} = -(2.0\times10^{-3}\,H)\left(2.0\frac{A}{s}\right) = -4.0\times10^{-3}\,V$$

Note: The negative sign is an indication of the direction of the induced voltage. Due to the nature of the induced emf, the negative sign means that it will oppose the increasing current change and produce an emf in the opposite direction of the external emf source creating the current.

Inductance

Energy Stored in an Inductor

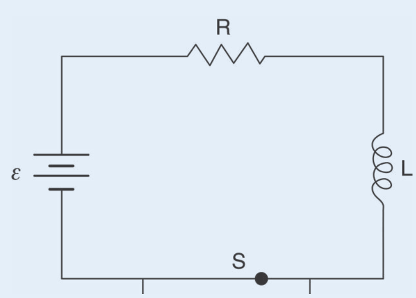
Since the inductor opposes the changes in the circuit, when the inductor resists the original current increase in a circuit (turning on a circuit) it takes extra energy to put into the circuit to reach the final steady state. This extra energy is given back to the circuit when the current is decreasing (circuit being turned off). This energy can be considered stored in the magnetic field of the inductor. The definition of this energy is

$$U_L = \frac{1}{2}LI^2$$

where I is the magnitude of the current while the circuit with an inductor is in a steady-state condition (not changing).

Example

A circuit is shown below with a voltage source, a resistor, and an inductor.



Determine the energy stored in the inductor after the circuit has reached steady state (i.e., the switch S has been closed for a very long time). The inductor has a value of $L = 2\ mH$.

$$\begin{aligned} EMF &= 10\ Volts \\ R &= 20\ ohms \\ L &= 2\ mH \end{aligned}$$

In order to determine the energy stored in the inductor we need to determine the current in the circuit. The current is determined very simply by using Ohm's law. The ideal inductor is considered to have zero resistance as it is simply a coiled wire. So the current in steady state conditions is simply:

$$I = \frac{\varepsilon}{R} = \frac{10V}{20\Omega} = 0.5A$$

Now we will apply the definition of energy in an inductor:

$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}(2\times10^{-3}H)(0.5A)^2 = 2.5\times10^{-4}J$$

This is a very small amount of energy! So the battery source needed to provide an extra 0.25 mJ of energy to transfer into the circuit to get to steady state condition, and when the circuit switch is opened the current will not immediately go to zero — it will exponentially decay to zero. That extra time that the current persists needs energy to move the charge, and the 0.25 mJ is returned back to the system in the form of a current that persists a little bit longer after the switch is opened.